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TURBULENT MIXING OF THREE
CONCENTRIC REACTING JETS
Part I

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TURBULENT MIXING OF THREE CONCENTRIC
REACTING JETS

Part I

By Beverly J. Audeh
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SUMMARY

22812

An analytical method is outlined for the calculation of mixing and combustion characteristics of a rocket exhaust jet with turbine gases exhausting peripherally about the nozzle exit and with a surrounding external air stream. This analytical method is currently being programmed on a digital computer.

Audeh

INTRODUCTION

The exhaust gases of fuel-rich liquid propellant rockets usually undergo an afterburning secondary combustion process upon mixing with the external flow. The afterburning is enhanced by the addition of fuel-rich gases from the turbine driven pumps exhausting around the hot gases leaving the expansion nozzle.

The turbulent mixing of supersonic jets has been treated

in various ways as the subject of several separate studies. Libby (Ref. 1) analyzed the mixing flow field for supersonic combustion studies which assumed equilibrium chemistry. An analysis was made by Vasiliu (Ref. 2) of the mixing of a rocket exhaust jet and a supersonic air stream assuming non-equilibrium chemistry. Emmons (Ref. 3) treated both the equilibrium and frozen case in supersonic flow. A solution using the flame front principle in place of equilibrium chemistry was formulated by Rozsa (Ref. 4).

Heat Technology Laboratory is presently developing a computer program to calculate mixing and combustion characteristics of a coaxial rocket exhaust jet. This jet is assumed to be composed of fuel rich turbine gases exhausting peripherally about the hot rocket exhaust gases and surrounded by an external air stream. This computer program will assume axial symmetry of the three coaxial gas streams and will provide for an arbitrary radial distribution of initial velocities and species mass fractions at the nozzle exit plane. The analytical method, which is presently being programmed for a computer is described in this report.

LIST OF SYMBOLS

a_i	Thermodynamics constants
c_k	Mass fractions of the k th element
c_p	Specific heat, cal/g°K
\bar{c}_q	Mass fractions of the q th species
D_{eff}	Coefficient of effective diffusion, cm ² /sec
F	Symbolic of a general function
$f(Z)$	Arbitrary function representing initial profiles
H	Total enthalpy, cal/g
h	Static enthalpy (intermediate value defined by Eq. (9)), cal/g
h'	Static enthalpy (defined by Eq. (10)), cal/g
$\Delta h_{q,0}$	Heat of formation of species q at T_0 , cal/g
I_0	Modified Bessel function
k	Index
K_C	Equilibrium constant in terms of concentrations
K_P	Equilibrium constant in terms of partial pressures
l	Index
Le	Lewis number, $\rho D_{eff} c_p / \kappa$, dimensionless
M	Molecular weight, g/mole
n	Number of species within a particular reaction
P	P function, Eq. (13)
p	Pressure, dyne/cm ²
Pr	Prandtl number, $c_p \mu / \kappa$, dimensionless
q	Index

R	Universal gas constant
r	Radius, cm
s	Maximum number of species
T	Temperature, °K
U	Velocity in the x direction, cm/sec
V	Velocity in the r direction, cm/sec
W	Mass rate of reaction, g/sec cm ³
x	Distance in axial direction, cm
Z	ψ/ψ_e
$\alpha'_{ql}, \alpha''_{ql}$	Stoichiometric coefficient of the q th species in the l th reaction, reactants and products respectively
ϵ	Coefficient of eddy viscosity, g/cm sec
κ	Thermal conductivity, cal/sec cm °K
μ	Dynamic viscosity, g/cm sec
ξ	Transformation variable defined by Eq. (16)
ρ	Density, g/cm ³
ψ	Dimensionless stream function defined by Eq. (4)

Subscripts:

e	Indicates external conditions
eff	Indicates effective quantity
i	Index
j	Indicates jet conditions
k	Index
l	Index
max	Maximum
min	Minimum
q	Index

t Indicates turbulent conditions

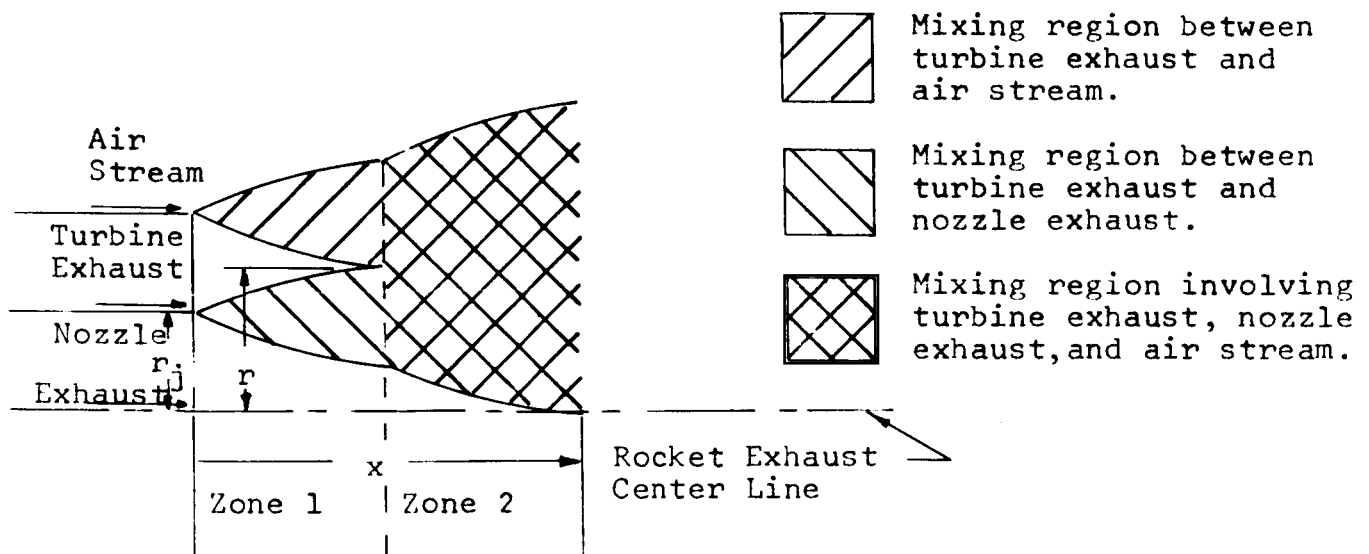
Superscripts:

* Dimensionless quantity

DISCUSSION

Mathematical Model

A sketch of the mathematical model of the rocket exhaust, turbine exhaust, and the surrounding air stream is shown as follows:



The calculation of the mixing and combustion characteristics of the flow model involves two zones which are treated separately. In zone 1, mixing occurs: (1) between the turbine exhaust and the nozzle exhaust and (2) between the turbine exhaust and the air stream. These two mixing processes are treated independently without mutual interaction. The two mixing processes overlap at the interface between zones 1 and 2, after which the mixing process involves all three gas streams. Calculation of the two mixing cases in zone 1

provides profiles of velocity and mass fractions at the interface between zones 1 and 2. These profiles are then used as initial conditions for mixing calculations of all three gas streams in zone 2.

Flow Field Equations

The flow field equations for axisymmetric turbulent jet mixing with chemical reactions are derived in Appendix I and are given as follows:

$$\frac{\partial U^*}{\partial x^*} = \epsilon^* \frac{1}{\psi} \frac{\partial}{\partial \psi} \left(\psi \frac{\partial U^*}{\partial \psi} \right) \quad (1)$$

$$\frac{\partial H}{\partial x^*} = \epsilon^* \frac{1}{\psi} \frac{\partial}{\partial \psi} \left(\psi \frac{\partial H}{\partial \psi} \right) \quad (2)$$

$$\frac{\partial \bar{c}_q}{\partial x^*} = \epsilon^* \frac{1}{\psi} \frac{\partial}{\partial \psi} \left(\psi \frac{\partial \bar{c}_q}{\partial \psi} \right) + \frac{r_j}{\rho_j U_j} \frac{W_q}{\rho^* U^*} \quad (3)$$

where the dimensionless stream function ψ is defined by

$$\begin{aligned} \psi \frac{\partial \psi}{\partial r^*} &= \left(\frac{\rho_e U_e}{\rho_j U_j} \right) \rho^* U^* r^* \\ \psi \frac{\partial \psi}{\partial x^*} &= - \left(\frac{\rho_e U_e}{\rho_j U_j} \right) \rho^* V^* r^* \end{aligned} \quad (4)$$

and

$$U^* = \frac{U}{U_j}, \quad V^* = \frac{V}{U_j}, \quad x^* = \frac{x}{r_j}, \quad r^* = \frac{r}{r_j},$$

$$\epsilon^* = \frac{\epsilon}{\rho_j U_j r_j}, \quad \rho^* = \frac{\rho}{\rho_j}.$$

Also, x and r are the axial and radial distances respectively; U and V are the velocity components in the axial and radial directions respectively; ϵ is the coefficient of eddy viscosity; H is the total enthalpy; \bar{c}_q is the mass fraction of the q th species; ρ is the density; W_q is the mass rate of reaction per unit volume of the q th species; and the subscripts j and e refer to the rocket exhaust jet exit conditions and the external free stream conditions respectively.

In addition to the above equations, the following equations which govern the reactions of the various chemical species are required.

Stoichiometric Restraints. - The fact that the mass rate of change of any of the elements is zero is expressed by equations of the following form:

$$W_k = \left\{ \sum_{q=1}^s \sum_{\ell=1}^r \frac{\alpha'_{\ell q k} M_k W_{\ell q k}}{\alpha''_{\ell q k} M_{q k}} \right\}_{k=\text{constant}} = 0 \quad (5)$$

where W_k is the mass rate of change per unit volume of the k th element, M_k and M_{qk} are the atomic weight of the element

k and the molecular weight, respectively, of the q th species containing element k; α''_{lqk} and α'_{lqk} are the stoichiometric coefficients for the l th reaction and the q th species containing the k th element of the products and reactants respectively; W_{lqk} is the mass rate of change per unit volume of the q th species containing the k th element for the l th reaction; r is the total number of reactions; and s is the total number of species.

The mass fraction of the k th element, c_k , is expressed as

$$c_k = \left\{ \sum_{q=1}^s \sum_{l=1}^r \frac{\alpha'_{lqk} M_k}{\alpha''_{lqk} M_{lqk}} \bar{c}_{lqk} \right\}_{k=\text{constant}} \quad (6)$$

For one step reactions involving s species with r reactions, there are s - r elements involved (Ref. 5). There are, therefore, s - r relationships for the mass fractions of the elements.

Equilibrium Relations. - The assumption of chemical equilibrium provides additional restraints described by the equilibrium relations for the chemical reactions. The equilibrium equations are of the form (Ref. 6)

$$K_{c_l} = \rho \sum_{q=1}^r (\alpha''_{ql} - \alpha'_{ql}) \prod_{q=1}^n \left(\frac{\bar{c}_{ql}}{M_{ql}} \right)^{(\alpha''_{ql} - \alpha'_{ql})} \quad (7)$$

where K_{c_ℓ} is the equilibrium constant for the ℓ th reaction, $\bar{c}_{q\ell}$ is the unknown mass fraction of the q th species in the ℓ th reaction, $\alpha''_{q\ell}$ and $\alpha'_{q\ell}$ are the stoichiometric coefficients of the products and reactants for the q th species in the ℓ th reaction, and ρ is the density. The equilibrium constant K_{c_ℓ} is a function of T only since in this case no pressure gradient exists. The equilibrium relations provide r equations but introduce the density, ρ , and the temperature, T , as unknowns. Combined with the $s - r$ stoichiometric relationships there are now $(s - r) + r = s$ equations and $s + 2$ unknowns.

Equation of State. - The relationship between the density and the unknown mass fractions from the equation of state is

$$\frac{\rho}{\rho_j} = \frac{T_j}{T} \frac{\left(\sum_{q=1}^s \frac{\bar{c}_q}{M_q} \right)_j}{\left(\sum_{q=1}^s \frac{\bar{c}_q}{M_q} \right)} \quad (8)$$

for a constant pressure case. This provides a set of $s + 1$ equations with $s + 2$ unknowns.

Static Enthalpy. - The static enthalpy relations will complete the necessary set of equations to solve for the unknown species mass fractions, density, and temperature. The

static enthalpy, h , in terms of species mass fractions is

$$h = \sum_{q=1}^s \bar{c}_q \int_{T_0}^T c_{p,q} dT + \sum_{q=1}^s \bar{c}_q \Delta h_{q,0} \quad (9)$$

where $\Delta h_{q,0}$ is the heat of formation of the q th species at the reference temperature T_0 and the static enthalpy h becomes an additional unknown.

Another equation for static enthalpy, h' , may be obtained through consideration of the total enthalpy H defined by

$$H = h' + \frac{1}{2} U_e^2 U^*{}^2 \quad (10)$$

In Eq. (9), h represents an intermediate value obtained iteratively. It is dependent upon temperature and species mass fractions and is compared with the value h' obtained from Eq. (10) as a determination of the accuracy of the iteration.

With Eqs. (9) and (10) a set of $s + 3$ necessary equations are sufficient to solve for the $s + 3$ unknowns when $h = h'$.

Solution of Equations

Equations (1) and (2) have the form of the classical diffusion or heat conduction equation which for this case has the following solutions (Ref. 7):

$$U^* = 1 - P \left(1 - \frac{U_j}{U_e} \right) \quad (11)$$

and

$$H = H_e \left[1 - P \left(1 - \frac{H_j}{H_e} \right) \right] \quad (12)$$

The P represents the P function described in Carslaw and Jaeger (Ref. 8) as:

$$P = (\psi_j/2\xi) \int_0^1 \exp \left\{ - \left[\frac{(\psi/\psi_j)^2 + Z^2}{4\xi/\psi_j} \right] \right\} I_0(\psi Z/2\xi) f(Z) Z dZ \quad (13)$$

where $I_0(\psi Z/2\xi)$ is the modified Bessel function defined by:

$$I_0(\psi Z/2\xi) = 1 + \frac{(\psi Z/2\xi)^2}{2^2} + \frac{(\psi Z/2\xi)^4}{2^2 4^2} + \dots \quad (14)$$

The variable of integration Z is defined by:

$$Z = \frac{\psi}{\psi_e} \quad (15)$$

where ψ_e is the stream function defining the edge of the jet mixing region and corresponding to free stream conditions.

The transformation variable ξ is defined by:

$$\xi = \int_0^{x^*} \epsilon^* dx^* \quad (16)$$

Equation (3) may be placed in the same form as Eqs. (1) and (2) through use of the chemical reaction equations. This is accomplished by a procedure outlined by Rozsa in Ref. 4 and described in the following paragraphs.

Multiplication of Eq. (3) by the constant $\left(\frac{\alpha'_{lqk} M_k}{\alpha''_{lqk} M_{lqk}} \right)$

gives

$$\frac{\partial}{\partial x^*} \left[\left(\frac{\alpha'_{lqk} M_k}{\alpha''_{lqk} M_{lqk}} \right) \bar{c}_q \right] = \epsilon^* \frac{1}{\psi} \frac{\partial}{\partial \psi} \left[\psi \frac{\partial \left(\frac{\alpha'_{lqk} M_k}{\alpha''_{lqk} M_{lqk}} \right) \bar{c}_q}{\partial \psi} \right] + \frac{r_j}{\rho_j U_j \rho^* U^*} \left(\frac{\alpha'_{lqk} M_k}{\alpha''_{lqk} M_{lqk}} \right) W_{lqk} \quad (17)$$

Summing s - r of the modified species continuity equations (Eq. (17)), the following equation is obtained.

$$\sum_{q=1}^s \sum_{l=1}^r \frac{\partial \left[\left(\frac{\alpha'_{lqk} M_k}{\alpha''_{lqk} M_{lqk}} \right) \bar{c}_q \right]}{\partial x^*} = \sum_{q=1}^s \sum_{l=1}^r \epsilon^* \frac{1}{\psi} \frac{\partial}{\partial \psi} \left[\psi \frac{\partial \left(\frac{\alpha'_{lqk} M_k}{\alpha''_{lqk} M_{lqk}} \right) \bar{c}_q}{\partial \psi} \right] + \sum_{q=1}^s \sum_{l=1}^r \frac{r_j}{\rho_j U_j \rho^* U^*} \left(\frac{\alpha'_{lqk} M_k}{\alpha''_{lqk} M_{lqk}} \right) W_{lqk} \quad (18)$$

The summation signs may be moved inside the partial derivatives for a continuous function giving

$$\frac{\partial \left[\sum_{q=1}^s \sum_{l=1}^r \left(\frac{\alpha'_{lqk} M_k}{\alpha''_{lqk} M_{lqk}} \right) \bar{c}_{lqk} \right]}{\partial x^*} = \epsilon^* \frac{1}{\psi} \frac{\partial}{\partial \psi} \left[\psi \frac{\partial \left(\sum_{q=1}^s \sum_{l=1}^r \frac{\alpha'_{lqk} M_k}{\alpha''_{lqk} M_{lqk}} \bar{c}_{lqk} \right)}{\partial \psi} \right] + \frac{r_j}{\rho_j U_j \rho^* U^*} \sum_{q=1}^s \sum_{l=1}^r \left(\frac{\alpha'_{lqk} M_k}{\alpha''_{lqk} M_{lqk}} \right) W_{lqk} \quad (19)$$

Substituting values from Eq. (5) and (6) into Eq. (19), the form of the equation becomes

$$\frac{\partial c_k}{\partial x^*} = \epsilon^* \frac{1}{\psi} \frac{\partial}{\partial \psi} \left(\psi \frac{\partial c_k}{\partial \psi} \right) + \frac{r_j}{\rho_j U_j \rho^* U^*} W_k \quad (20)$$

But $W_k = 0$ (Eq. (5)), so Eq. (20) reduces to

$$\frac{\partial c_k}{\partial x^*} = \epsilon^* \frac{1}{\psi} \frac{\partial}{\partial \psi} \left(\psi \frac{\partial c_k}{\partial \psi} \right) \quad (21)$$

which is of the same form as Eqs. (1) and (2) and has the solution:

$$c_k = c_{k_e} \left[1 - P \left(1 - \frac{c_j}{c_{k_e}} \right) \right] \quad (22)$$

This result gives the chemical mass fractions in terms of the elements rather than the species. It is necessary, therefore, to obtain the species mass fractions by relating these results to the species mass fraction as determined by Eq. (6).

There are $s - r$ equations of c_k relating the s unknown species mass fractions. From Eq. (7), r more equations can be obtained but the unknowns have increased to $s + 2$. Introduction of Eq. (8) provides $s + 1$ equations relating the $s + 2$ unknowns. With Eq. (9) a set of $s + 2$ equations is evolved with $s + 3$ unknowns. Finally, Eq. (10) may be used

to provide a sufficient set of $s + 3$ equations involving $s + 3$ unknowns.

An example of the solution of the chemical equations is included to illustrate the calculation procedure. A system is assumed to contain the following species: O_2 , H_2 , H_2O , C , and CO_2 . This includes the elements O , H , and C .

The reactions involved are



and



In this example the number of species s is 5, the number of reactions r is 2, and the number of elements k is 3.

From the above reactions, the elemental mass fractions can be obtained using Eq. (6).

$$2c_H = \frac{(2)(2)}{(1)(2)} \bar{c}_{H_2} + \frac{(2)(2)}{(2)(18)} \bar{c}_{H_2O} \quad (25)$$

$$2c_O = \frac{(2)(32)}{(1)(32)} \bar{c}_{O_2} + \frac{(1)(32)}{(2)(18)} \bar{c}_{H_2O} + \frac{(1)(32)}{(1)(44)} \bar{c}_{CO_2} \quad (26)$$

$$c_C = \frac{(1)(12)}{(1)(12)} \bar{c}_C + \frac{(1)(12)}{(1)(44)} \bar{c}_{CO_2} \quad (27)$$

Reference values for the elemental mass fractions c_{k_j} and c_{k_e} are defined as values of c_k at the jet exit and in the external flow respectively.

Values for c_k can be obtained from Eq. (22). These c_k values can be expressed numerically in terms of the previously determined velocity ratios by solving for P from Eq. (11) and substituting it in Eq. (22):

$$c_k = c_{k_e} \left(\frac{U^* - \frac{U_j}{U_e}}{1 - \frac{U_j}{U_e}} \right) + c_{k_j} \left(\frac{1 - U^*}{1 - \frac{U_j}{U_e}} \right) \quad (28)$$

For the case being treated here Eq. (28) becomes

$$c_H = c_{H_e} \left\{ \frac{U^* - \frac{U_j}{U_e}}{1 - \frac{U_j}{U_e}} \right\} + c_{H_j} \left\{ \frac{1 - U^*}{1 - \frac{U_j}{U_e}} \right\} \quad (29)$$

or in expanded form

$$\begin{aligned} \bar{c}_{H_2} + \frac{1}{9} \bar{c}_{H_2O} = \\ \left(\bar{c}_{H_2} + \frac{1}{9} \bar{c}_{H_2O} \right)_e \left\{ \frac{U^* - \frac{U_j}{U_e}}{1 - \frac{U_j}{U_e}} \right\} + \left(\bar{c}_{H_2} + \frac{1}{9} \bar{c}_{H_2O} \right)_j \left(\frac{1 - U^*}{1 - \frac{U_j}{U_e}} \right) \end{aligned} \quad (30)$$

also

$$c_O = (c_O)_e \left\{ \frac{U^* - \frac{U_j}{U_e}}{1 - \frac{U_j}{U_e}} \right\} + (c_O)_j \left(\frac{1 - U^*}{1 - \frac{U_j}{U_e}} \right) \quad (31)$$

or in expanded form

$$\begin{aligned} \bar{c}_{O_2} + \frac{8}{9} \bar{c}_{H_2O} + \frac{8}{11} \bar{c}_{CO_2} = & \left(\bar{c}_{O_2} + \frac{8}{9} \bar{c}_{H_2O} + \frac{8}{11} \bar{c}_{CO_2} \right)_e \left\{ \frac{U^* - \frac{U_j}{U_e}}{1 - \frac{U_j}{U_e}} \right\} \\ & + \left(\bar{c}_{O_2} + \frac{8}{9} \bar{c}_{H_2O} + \frac{8}{11} \bar{c}_{CO_2} \right)_j \left\{ \frac{1 - U^*}{1 - \frac{U_j}{U_e}} \right\} \end{aligned} \quad (32)$$

Also

$$c_C = c_{C_e} \left\{ \frac{U^* - \frac{U_j}{U_e}}{1 - \frac{U_j}{U_e}} \right\} + c_{C_j} \left\{ \frac{1 - U^*}{1 - \frac{U_j}{U_e}} \right\} \quad (33)$$

or in expanded form

$$\begin{aligned} \bar{c}_C + \frac{3}{11} \bar{c}_{CO_2} = & \left(\bar{c}_C + \frac{3}{11} \bar{c}_{CO_2} \right)_e \left\{ \frac{U^* - \frac{U_j}{U_e}}{1 - \frac{U_j}{U_e}} \right\} + \left(\bar{c}_C + \frac{3}{11} \bar{c}_{CO_2} \right)_j \left\{ \frac{1 - U^*}{1 - \frac{U_j}{U_e}} \right\} \end{aligned} \quad (34)$$

This provides 3 equations (Eqs. (30), (32), and (34) in terms of the 5 unknown specie mass fractions, \bar{c}_{H_2} , \bar{c}_{H_2O} , \bar{c}_{O_2} , \bar{c}_C , and \bar{c}_{CO_2} .

Two additional equations are written using Eq. (7) (for which the stoichiometric coefficients are obtained from Eqs. (23) and (24)):

$$K_{C_1} = \rho^{[2-(2+1)]} \left\{ \left(\frac{\bar{c}_{H_2O}}{M_{H_2O}} \right)^2 \left(\frac{\bar{c}_{H_2}}{M_{H_2}} \right)^{-2} \left(\frac{\bar{c}_{O_2}}{M_{O_2}} \right)^{-1} \right\} \quad (35)$$

$$K_{C_2} = \rho^{[1-(1+1)]} \left\{ \left(\frac{\bar{c}_{CO_2}}{M_{CO_2}} \right)^1 \left(\frac{\bar{c}_C}{M_C} \right)^{-1} \left(\frac{\bar{c}_{O_2}}{M_{O_2}} \right)^{-1} \right\} \quad (36)$$

There are now 5 equations involving the 5 unknown species mass fractions, but two extra unknowns, density and the temperature dependent K_{C_l} , have been added.

A density relationship from Eq. (8) is

$$\rho = \frac{\rho_j T_j}{T} \frac{\left[\frac{\bar{c}_{H_2}}{M_{H_2}} + \frac{\bar{c}_{O_2}}{M_{O_2}} + \frac{\bar{c}_{H_2O}}{M_{H_2O}} + \frac{\bar{c}_C}{M_C} + \frac{\bar{c}_{CO_2}}{M_{CO_2}} \right]_j}{\left[\frac{\bar{c}_{H_2}}{M_{H_2}} + \frac{\bar{c}_{O_2}}{M_{O_2}} + \frac{\bar{c}_{H_2O}}{M_{H_2O}} + \frac{\bar{c}_C}{M_C} + \frac{\bar{c}_{CO_2}}{M_{CO_2}} \right]} \quad (37)$$

This provides 6 equations involving 7 unknowns.

An iteration procedure using T as the iterative parameter using Eqs. (30), (32), (34), (35), (36) and (37) is applied. Equations (9) and (10) are used to check the correctness of the assumed temperature values. These become

$$h' = H - \frac{1}{2} U^*{}^2 U_e^2 \quad (38)$$

and

$$\begin{aligned}
 h = & \left[\bar{c}_{H_2} \int_{T_o}^T c_{p_{H_2}} dT + \bar{c}_{O_2} \int_{T_o}^T c_{p_{O_2}} dT + \bar{c}_{H_2O} \int_{T_o}^T c_{p_{H_2O}} dT + \right. \\
 & \left. \bar{c}_C \int_{T_o}^T c_{p_C} dT + \bar{c}_{CO_2} \int_{T_o}^T c_{p_{CO_2}} dT \right] + \left[\bar{c}_{H_2} \Delta h_{H_2,o} + \right. \\
 & \left. \bar{c}_{O_2} \Delta h_{O_2,o} + \bar{c}_{H_2O} \Delta h_{H_2O,o} + \bar{c}_C \Delta h_{C,o} + \bar{c}_{CO_2} \Delta h_{CO_2,o} \right]
 \end{aligned}
 \tag{39}$$

where T is the temperature assumed and used to solve for the unknown mass fractions; U^* and H are previously determined from Eqs. (11) and (12); and U_e is the velocity of the external air. Whenever the static enthalpy values agree, that is

$$|h - h'| < \text{maximum allowable error} \tag{40}$$

the assumed T value is taken as correct and the iteration procedure is completed.

Transformations. - The foregoing calculation procedure provides a solution in terms of the transformation variables ξ and ψ . A transformation of these variables must be made to obtain the solution in terms of the physical coordinates x and r .

The physical coordinate x is formed in terms of the transformation variable ξ by integrating Eq. (16). In order to perform this integration, it is necessary to relate the coefficient of eddy viscosity ϵ to the x coordinate. As yet there is no complete agreement among investigators regarding this relationship; however, this analysis employs the following relation suggested by Kleinstein (Ref. 7) and used by Rozsa (Ref. 4):

$$\epsilon = 0.00075 \times [(\rho U)_{\max} - (\rho U)_{\min}] \quad (41)$$

This relation is applicable in a region where the mixing layer has not spread across the entire jet plume. Using the relation of Eq. (41), Eq. (16) may be integrated to yield:

$$x = r_j \left[\frac{2\xi (\rho_j U_j)}{0.00075 [(\rho U)_{\max} - (\rho U)_{\min}]} \right]^{1/2} \quad (42)$$

The physical coordinate r is found in terms of the transformation variable ψ by integrating Eq. (4). This results in the following relationship:

$$r^2 = r_j^2 \left(\frac{\rho_j U_j}{\rho_e U_e} \right) \int_0^\psi \frac{d\psi^2}{\rho^* U^*} \quad (43)$$

Development of Computer Program

A computer program is being constructed to solve separately the case for mixing of turbine exhaust and external

air and the case for mixing of turbine exhaust and rocket exhaust gases. The results of these two cases will be used to treat the combined mixing of external air, turbine exhaust, and rocket exhaust. A simplified flow diagram of the program is given in Appendix II. Upon completion of this program, computer runs will be made for cases approximating the afterburning of H-1 engine flows at low altitudes. The output will be profiles for the velocity, temperature, and species mass fractions plotted as a function of position. Comparisons of the results of this study will be made with all available data from afterburning of turbine exhausts.

CONCLUSIONS

This study is an extension of free jet mixing analyses that use the P function solution. Previous studies reported in the literature have applied the P function solution to two coaxial, axisymmetric jet streams with uniform initial conditions. As a result of this study, the P function has been modified to allow arbitrary initial profiles which makes it applicable for use within intersecting mixing regions. As reported here, the modified P function is applied to the turbulent mixing of three coaxial, axisymmetric jets of reactive gases. A computer program presently under development will provide profiles for the velocity, temperature, and species mass fractions as a function of position. The program will use the modified P

function such that both uniform and non-uniform initial conditions may be introduced.

APPENDIX I

SIMPLIFICATION OF FLOW EQUATIONS

The flow equations describing turbulent mixing of reacting gases may be simplified by the procedure outlined below.

Flow Equations

The following flow equations (Ref. 9) describe steady, turbulent, axisymmetric, reacting gas flow:

Conservation of Mass

$$\frac{\partial(\rho U r)}{\partial x} + \frac{\partial(\rho V r)}{\partial r} = 0 \quad (\text{I-1})$$

Conservation of Momentum

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial r} = - \frac{\partial p}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left[(U + \epsilon) r \frac{\partial U}{\partial r} \right] \quad (\text{I-2})$$

Conservation of Energy

$$\begin{aligned}
 \rho U \frac{\partial H}{\partial x} + \rho V \frac{\partial H}{\partial r} &= \frac{1}{r} \frac{\partial}{\partial r} \left(\left[\frac{\mu}{Pr} + \frac{\epsilon}{Pr_t} \right] r \frac{\partial H}{\partial r} \right) + \\
 \frac{1}{r} \frac{\partial}{\partial r} &\left(\left[\frac{\mu}{Pr} (Le - 1) + \frac{\epsilon}{Pr_t} (Le_t - 1) \right] r \sum_{q=1}^n h_q \frac{\partial \bar{c}_q}{\partial r} \right) + \\
 \frac{1}{r} \frac{\partial}{\partial r} &\left\{ \left[\mu \left(1 - \frac{1}{Pr} \right) + \epsilon \left(1 - \frac{1}{Pr_t} \right) \right] r U \frac{\partial U}{\partial r} \right\} \quad (I-3)
 \end{aligned}$$

Conservation of Species

$$\begin{aligned}
 \rho U \frac{\partial \bar{c}_q}{\partial x} + \rho V \frac{\partial \bar{c}_q}{\partial r} &= \frac{1}{r} \frac{\partial}{\partial r} \left(\left[\rho D_{eff} + \right. \right. \\
 \left. \left. \rho D_{eff,t} \right] r \frac{\partial \bar{c}_q}{\partial r} \right) &+ w_q \quad (I-4)
 \end{aligned}$$

The above equations may be simplified through the use of a number of assumptions: the dynamic viscosity μ is negligible compared to the eddy viscosity ϵ ; and the turbulent Lewis and Prandtl numbers Le_t and Pr_t are unity with the result that the product $\rho D_{eff,t}$ is equal to the eddy viscosity ϵ . Eqs. (I-1-4) are therefore simplified to

Conservation of Mass

$$\frac{\partial(\rho U r)}{\partial x} + \frac{\partial(\rho V r)}{\partial r} = 0 \quad (I-5)$$

Conservation of Momentum

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial r} = - \frac{\partial p}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left(\epsilon r \frac{\partial U}{\partial r} \right) \quad (\text{I-6})$$

Conservation of Energy

$$\rho U \frac{\partial H}{\partial x} + \rho V \frac{\partial H}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left(\epsilon r \frac{\partial H}{\partial r} \right) \quad (\text{I-7})$$

Conservation of Species

$$\rho U \frac{\partial \bar{c}_q}{\partial x} + \rho V \frac{\partial \bar{c}_q}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left(\epsilon r \frac{\partial \bar{c}_q}{\partial r} \right) + W_q \quad (\text{I-8})$$

The above Eqs. (I-5-8) may be reduced to the form of Eqs. (1-3) in the text by the introduction of the von Mises transformation (Ref. 10) defined by Eq. (4) and the introduction of the non-dimensional parameters denoted by asterisks.

APPENDIX II

EQUATIONS FOR JET MIXING PROGRAM

An outline of the input required for the jet mixing computer program, the equations involved, and the resultant output is made here. All of the symbols are defined in the list of symbols.

The following input are required: a_{1q} , R , U_j , U_e , ξ_{\max} , ψ_{\max} , H_e , H_j , K_{pl} , α'_{ql} , α''_{ql} , s , M_{qlk} , $(\bar{c}_{qlk})_e$, $(\bar{c}_{qlk})_j$, ρ_e , T_e , T_o , r_j , and $f(Z)$.

Thermodynamic values for $c_{p,q}$ and $\Delta h_{q,o}$ are initially calculated from

$$c_{p,q} = R (a_{1q} + a_{2q} T + a_{3q} T^2 + a_{4q} T^3 + a_{5q} T^4) \quad (\text{II-1})$$

and

$$\Delta h_{q,o} = R (a_{1q} T + \frac{a_{2q}}{2} T^2 + \frac{a_{3q}}{3} T^3 + \frac{a_{4q}}{4} T^4 + \frac{a_{5q}}{5} T^5 + a_{6q}) \quad (\text{II-2})$$

For given ξ and ψ values, U^* is evaluated using

$$P = \left\{ \frac{1 - U^*}{1 - \frac{U_j}{U_e}} \right\} = \frac{\psi_j}{2\xi} e^{-\frac{\psi^2}{4\xi}} \int_0^1 e^{-(Z^2/4\xi/\psi_j)} \left[\frac{\psi Z}{2\xi} \right] f(Z) Z dZ \quad (\text{II-3})$$

then

$$U^* = 1 - P \left(1 - \frac{U_j}{U_e} \right) \quad (\text{II-4})$$

and

$$U = U^* U_e \quad (\text{II-5})$$

The static enthalpy h' (controlled by diffusion) is evaluated as follows:

$$H = \frac{H_e \left(U^* - \frac{U_j}{U_e} \right) + H_j (1 - U^*)}{\left(1 - \frac{U_j}{U_e} \right)} \quad (\text{II-6})$$

and

$$h' = H - \frac{1}{2} (2.3901 \times 10^{-8})(U^2) \quad (\text{II-7})$$

The equilibrium constants in terms of concentrations are determined from

$$K_{C_\ell} = K_{P_\ell} (RT)^{\sum_{q=1}^n (\alpha'_{q\ell} - \alpha''_{q\ell})} \quad (\text{II-8})$$

These equilibrium constants are used to establish equilibrium relations.

$$K_{C_\ell} = \rho \sum_{q=1}^n (\alpha''_{q\ell} - \alpha'_{q\ell}) \prod_{q=1}^n \left(\frac{\bar{c}_{q\ell}}{M_{q\ell}} \right)^{(\alpha''_{q\ell} - \alpha'_{q\ell})} \quad (\text{II-9})$$

The elemental continuity equations are as follows:

$$\begin{aligned}
 & \left[\bar{c}_{1\ell k} + \sum_{q=2}^s \frac{\alpha'_{1\ell k} M_{1\ell k}}{\alpha''_{q\ell k} M_{q\ell k}} \bar{c}_{q\ell k} \right] = \\
 & \left(\bar{c}_{1\ell k} + \sum_{q=2}^s \frac{\alpha'_{1\ell k} M_{1\ell k}}{\alpha''_{q\ell k} M_{q\ell k}} \bar{c}_{q\ell k} \right)_e \left\{ \frac{U^* - \frac{U_j}{U_e}}{1 - \frac{U_j}{U_e}} \right\} + \\
 & \left(\bar{c}_{1\ell k} + \sum_{q=2}^s \frac{\alpha'_{1\ell k} M_{1\ell k}}{\alpha''_{q\ell k} M_{q\ell k}} \bar{c}_{q\ell k} \right)_j \left\{ \frac{1 - U^*}{1 - \frac{U_j}{U_e}} \right\} \Bigg|_k = \text{constant} \quad (\text{II-10})
 \end{aligned}$$

The density is related to the unknown mass fractions by

$$\rho = \frac{\rho_e T_e}{T} \frac{\left(\sum_{q=1}^s \frac{\bar{c}_q}{M_q} \right)_e}{\left(\sum_{q=1}^s \frac{\bar{c}_q}{M_q} \right)} \quad (\text{II-11})$$

The set of equations, Eqs. (II-9-11), provides a group of $s + 1$ equations with $s + 2$ unknowns where s is the number of species. A value of T is assumed. From Eq. (II-8) the equilibrium constants K_{c_ℓ} are evaluated. The unknown mass fractions, \bar{c}_q , are then determined along with a density ρ . To determine if the choice of T was correct, an additional enthalpy equation is introduced.

$$h = \sum_{q=1}^s \bar{c}_q \int_{T_0}^T c_{p,q} dT + \sum_{q=1}^s \bar{c}_q \Delta h_{q,0} \quad (\text{II-12})$$

Whenever $|h' - h|$ is less than a specified maximum allowable error, the \bar{c}_q values and the ρ values are taken as correct. Then ψ_i is replaced by ψ_{i+1} while ξ_i remains ξ_i until all ψ_i cases are evaluated for a given ξ_i . In the next step ξ_{i+1} replaces ξ_i and calculation for the \bar{c}_q values and ρ values for all the ψ_i cases are repeated.

Values of \bar{c}_q , T, and U in terms of ψ and ξ are printed out at this point.

Evaluation of r^* and x^* is made by

$$r^* = \left[\left(\frac{\rho_j U_j}{\rho_e U_e} \right) \int_0^{\psi} \frac{d(\psi^2)}{\rho^* U^*} \right]^{1/2} = \left[\left(\frac{\rho_j U_j}{\rho_e U_e} \right) \frac{\psi^2}{\rho^* U^*} \right]^{1/2} \quad (\text{II-13})$$

and

$$x^* = \left[\frac{2\xi \left(\rho_j U_j \right)}{0.00075 [(\rho U)_{\max} - (\rho U)_{\min}]} \right]^{1/2} \quad (\text{II-14})$$

From r^* and x^* , r and x can be determined by

$$r = r_j r^* \quad (\text{II-15})$$

and

$$x = r_j x^* \quad (\text{II-16})$$

Print out of \bar{c}_q , T, and U values in terms of r and x completes the program.

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